1 Pfaffian System and Involutivity

Let M be a manifold. Pfaffian system (I, J) with independence condition J satisfies the following conditions on M:

$$\theta^\alpha = 0, \quad \alpha = 1, \dots, s,$$

$$\Omega = \omega^1 \wedge \cdots \wedge \omega^p \neq 0$$

where $I = \text{span}\{\theta^1, \dots, \theta^s\}$ and $J = \text{span}\{\omega^1, \dots, \omega^p\}$.

Definition 1.1. For each $q = 1, \dots, p$, a **q-dimensional integral manifold** is a manifold N of dimension q such that $\theta^\alpha|_N = 0,\ \alpha = 1, \dots, s$, which implies that $d\theta|_N = 0$, so that $\phi|_N = 0$, for any ϕ in $\mathfrak{I}$, where $\mathfrak{I}$ is the differential ideal generated by θ^α's.

Definition 1.2. An **integral element** (x, E) of dimension q is a q-dimensional subspace E of T_xM such that $\phi|_E = 0$ for any $\phi \in \mathfrak{I}$. $V_q(\mathfrak{I})$ is defined to be the set of all q-dimensional integral elements for $q = 1, \dots, p-1$. $V(I, J)$ is defined to be the set of p-dimensional integral elements (x, E) such that $\Omega|_E \neq 0$.

Let (x, E) be a q-dimensional integral element of $\mathfrak{I}$. Let $e_1, \dots, e_q$ be a basis of E. Its **polar equations** are linear equations for the subspace of all $v \in T_xM$ such that $< \phi(x), e_1 \wedge \cdots \wedge e_q \wedge v >= 0$ for any $(q+1)$-form ϕ in $\mathfrak{I}$.

Definition 1.3. An integral element (x, E) is **K-regular** if (x, E) is a smooth point of $V_q(\mathfrak{I})$ and the rank of the polar equations is constant near (x, E). A **regular flag** of E is a sequence of K-regular elements such that $(0) = E^0 \subset E^1 \subset \cdots \subset E^{p-1} \subset E^p = E$.

Definition 1.4. A Pfaffian system (I, J) is **involutive** if a general integral element $(x, E) \in V(I, J)$ admits a regular flag.

Theorem 1.5 (Cartan-Kähler, [?]). *Let M be a C^ω manifold and $\mathfrak{I}$ be a C^ω differential ideal. Suppose that $(x, E) \in V(I, J)$ has a regular flag. Then there exists a C^ω integral manifold N of dimension p passing through x with $T_xN = E$. In other words, if C^ω Pfaffian system is involutive, it has C^ω solutions.*

If the Pfaffian system (I, J) is involutive, it does not imply any additional equations and the map $V(I, J) \to M$ is surjective or it satisfies the Frobenius condition $d\theta^\alpha = 0 \mod \theta$. Even if the map $V(I, J) \to M$ is surjective, still it may happen that the Pfaffian system implies additional equations.

Definition 1.6. Let (I, J) be a Pfaffian system with independence condition on M such that the map $V(I, J) \to M$ is surjective. Let

$$M^{(1)} = V(I, J) \subset G_pM.$$

Define the **first prolongation** $(I^{(1)}, J^{(1)})$ to be the restriction to $M^{(1)}$ of the canonical Pfaffian system of G_pM. Define $(I^{(q)}, J^{(q)})$ to be the first prolongation of $(I^{(q-1)}, J^{(q-1)})$ inductively for $q = 1, 2, \ldots$.

Cartan and Kuranishi showed it can happen that $(I^{(1)}, J^{(1)})$ is involutive while (I, J) is not.

Theorem 1.7 (Cartan-Kuranishi, [?]). *Given a Pfaffian system (I, J), there exists q_0 such that for $q \geq q_0$, $(I^{(q)}, J^{(q)})$ are involutive (under a mild regularity assumption).*

References

[1] R. Bryant, S. S. Chern, R. Gardner, H. Goldschmidt and P. Griffiths, *Exterior differential systems* (1991), Springer-Verlag, New York, Berlin, Heidelberg.

[2] R. Bryant, P. Griffiths and D. Yang, *Characteristics and existence of isometric embeddings*, Duke Math. J. **50** (1983), 893-994.